A Simple Macroeconomic Model with Default*

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Abstract

We find evidence of a statistical significant relation between credit spreads and other macroeconomic variables. In addition, we find presence of asymmetries in most of these variables. To address these facts, we develope a tractable model that admits a global solution and can account for some of these observations. In particular, it can generate the asymmetry in credit spreads as well as asymmetry in output responses to shocks.

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1 Introduction

Over the last two decades, starting with Bernanke and Gertler (1989), the literature in macroeconomic models with financial frictions grew exponentially. The work done by Kiyotaki and Moore (1997), Carlstrom and Fuerst (1997) and Bernanke, Gertler, and Gilchrist (1999), among others, shows how an economy subject to small shocks can experience large fluctuacions due to the presence of financial frictions: what is generally referred as the "financial accelerator".

Although models that include some of these features have become increasingly popular, a final agreement in a benchamrk framework is still waiting[1]. In this sense, there are, at least, two related issues. On one hand, these economies typically include a seizable amount of heterogeneity which in turn implies an infinite dimensional state space. Thus, exact solutions are unfeasible and approximation methods must be used [2]. On the other hand, and obviously related, many times assumptions are made so that a simpler solution method is accurate[3]. This generally accounts for assuming risk-neutrality among some agents which allows to reduce the dimensionality of the state space. One of the contributions of this paper is to tackle these issues by developing a model that admits an exact global solution without the need of linear assumptions on preferences.

The main findings of this paper are as follows. On the emprical side, we have three findings: (i) we find strong empricial evidence that links measures of financal frictions with other aggregate variables, (ii) we show that most of the variables we generally care in macroeconomics are far from being described by symmetric distributions, which provides incentives for our global solution methods, and (iii) we provide evidence that, contrary to other typical macroeconomic variables, credit spreads did not moderate during the last couple decades.

[1]See Gertler and Kiyotaki (2009)for an excellent discussion.

[2]For instance, the "bounded rationality" method propose by Krusell and Smith (1998). See Khan and Thomas. (2008) for a more recent discussion.

[3]Typically perturbation methods, for instance linear approximations

To address these facts, we develope a tractable model that admits a global solution and can account for some of these observations. In particular, it can generate the asymmetry in credit spreads as well as asymmetry in output responses to shocks.

Our model has two important agents: entrepreneurs and investors. The formers are the owners of the firms, they must invest in capital in order to produce output and they can fund themselves by issuing debt or by using their own wealth. Investors are the counterpart of this borrowing, they lend funds to entrepreneurs. However, returns on capital investment are stochatic and entrepreneurs can decide to default. Consequently, the price of the debt issued by entrepreneurs (the cost of funding) will reflect this default probability. The main mechanism in this paper is how entrepreneurs exploit the trade-off between issuing more debt versus obtaining a better price for that debt.

The model has three key ingredients. First, entrepreneurs will want to borrow from investors in order to build up capital and produce next period. However, and second, they can not commit to repay and thus the price of their debt reflects future default probabilities. Consequently, and third, an entrepreneur who wants to install a larger scale firm needs to borrow more, pushing up default probabilitites and thus making funding more expensive.

The combination of these ingredients imply a trade-off for the entrepreneur between investment and funding cost. An entrepreneur who is willing to borrow more and run a larger scale firm will have to compensate the increase in the default probability to the investor by issuing debt with a higher interest rate. The interaction is similar to the financial accelerator mechanism: after a bad shock hits the economy, (ceteris paribus) default probabilitites go up, which makes borrowing more expensive, disincentivize investment and thus further amplify the initial shock. Note however, that the effect could potentially be asymmetric: for an economy where default rates are not typically high, good shocks can not decrease default probabilties much while bad shocks will have larger absolute impact. In line with the emprical evidence, we show later that this is the case in our model.

The paper is organized as follows: section 2 provides some empirical evidence about the importance and recent behavior of variables related to financial frictions, section 3 describes the model and characterize the equilibrium, section 4 provides a simple example of the model, section 5 performes a quantitative evaluation and discuss the main results, section 6 concludes.

2 Empricial Evidence

In this section we show some evidence regarding financial frictions. Our focus will be in the measure of credit spreads constructed by Gilchrist and Zakrajsek (2012). One advantage of their computation is that it captures the idea of credit spreads that we use in our models: the difference in interest rates paid by a risky promise of chash flows compared to the *same* promise but without risk. In particular, for a large data set of corporate bonds, they compute the difference between observed price in secondary markets and what would the price of the same promise had been if made by the Treasury[4].

We divide the exposition in two: first, a simple look at data; and second, we show evidence from a vector autorregresive model with time-varying coefficients and stochastic volatilities as in Cogley and Sargent (2005).

2.1 A first look at data

Figure 1 shows the time series for GDP growth and credit spreads from 1973:1 to 2010:4. A few things stand out: first, GDP growth and credit spreads are negatively correlated; second, GDP growth is twice as volatile as credit spreads[5]; and third, credit spreads had a significant response during the last two recessions of 2001 and 2009.

[4]They need a model to do this pricing, see their paper for details.

[5]We computed standard deviations of variables divided by it mean to make comparisons fair

	spread	ΔY	ΔC	ΔI
Skewness	2.65	-0.31	-1.02	-0.88
Kurtosis	13.02	4.94	5.65	6.38

Table 1: Empirical Asymmetries

Figure 2 shows scatter plots (and the best linear fit) of credit spreads with typical macroeconomic variables: GDP growth, consumption growth, investment growth and 3 moths T-Bill interest rate, all in logs. For the first three variables the finding is in the same line as before: periods of higher credit spreads coincide with lower consumption and investment growth, being this particulary true for the later: correlations are -0.27 and -0.57 respectively. Interestingly as well, periods of high credit spreads are associated with lower interest rates; the sample correlation is -0.63.

Table 1 shows evidence of asymmetries for most of the typical variables we consider in macroeconomics. Growth rates for GDP, consumption and investment have negative skewness and kurtosis significantly higher than three. Thus, there is evidence of fat tails and an inclination to the left. For credit spreads the kurtosis is even higher although the skewness indicates an inclination to the right: periods of very high spreads. Figure 3 shows the histograms for each of these variables. This type of asymmetries can not be captured by linear methods and thus provides incentives to our solution method. As we show below, we can deliver some of these facts with a still tractable model.

Thus, from this simple exercise we observe that credit spreads has a statistical significant relation with variables that we are generally interested in macroeconomics. Consequently, defualt rates seems to be in the core of interesting business cycle fluctuations. Furthermore, we find evidence of asymmetries that suggest the need of global solution methods.

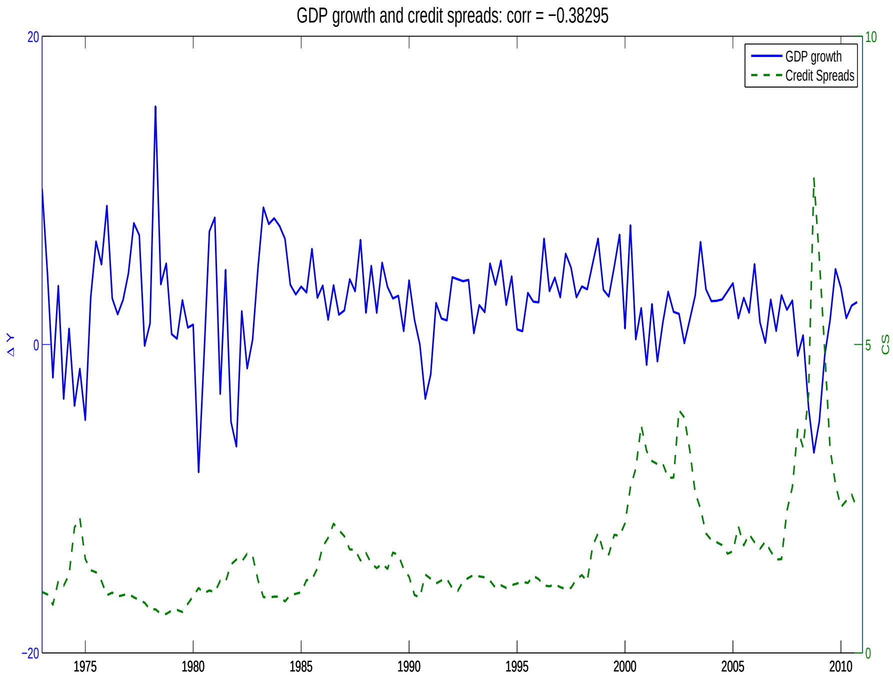

Figure 1: GDP Growth and Credit Spreads

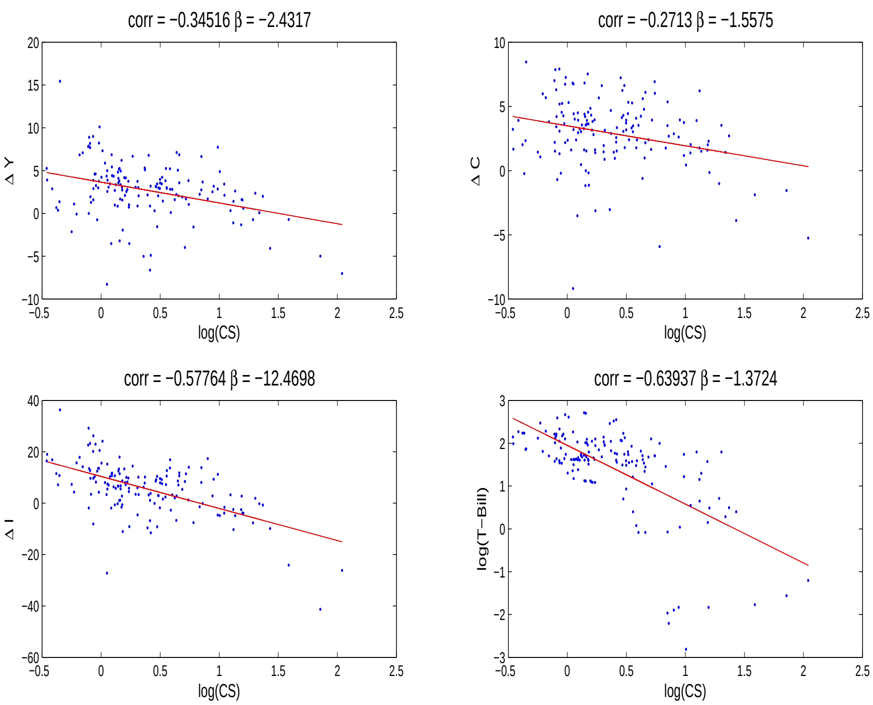

Figure 2: Scatter Pots with Credit Spreads

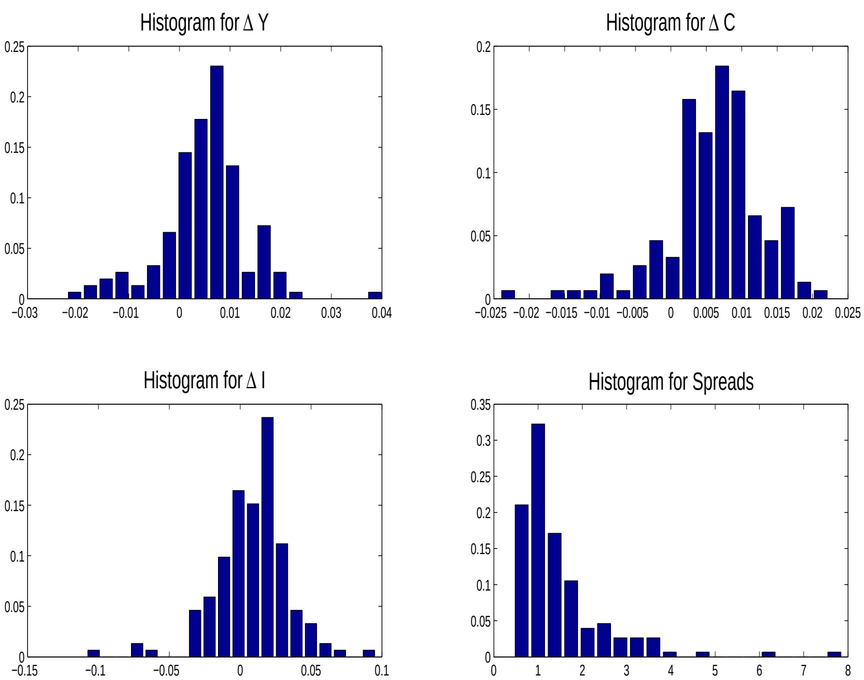

Figure 3: Empirical Histograms

2.2 Evidence from a time-varying VAR

2.2.1 Drifting Coefficients and Stochastic Volatility

In this section we use a very flexible statistical tool to understand better data. We are mostly interested in assesing how credit spreads evolved during the last couple decades. We will remain exposition short, but a more comprehensive exercise can be found in Navarro (2012). Our main finding is the following: while evidence of a great moderation is clear for most of our variables, this is not the case for credit spreads. The latter became more volatile than before, with a higher long run mean and particulary more reactive during recessions.

The model we are interested in estimating is as follows

$$y_t = X_t'\theta_t + R_t^{1/2}\xi_t \tag{1}$$

where y_t is an $n \times 1$ vector of (endogenous) variables, X_t is a matrix with ones and lagged values of y_t, and ξ_t is a vector of standard normal innovations. Finally, note that both autoregressive parameters θ_t as well as the covariance matrix R_t are allowed to move over time. In particular we assume that

$$\theta_t = \theta_{t-1} + v_t \tag{2}$$

$$\ln h_{it} = \ln h_{it-1} + \sigma_i \eta_{it} \tag{3}$$

where $R_t = B^{-1}H_tB^{-1\prime}$ and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_{21} & 1 & 0 & 0 \\ \vdots & \ddots & 1 & 0 \\ \beta_{n1} & \cdots & \beta_{nn-1} & 1 \end{pmatrix}, \quad H_t = \begin{pmatrix} h_{1t} & 0 & \cdots & 0 \\ 0 & h_{2t} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & h_{nt} \end{pmatrix} \tag{4}$$

Let θ^T, H^T and y^T be the history of the parameteres and data respectivelly

$$\begin{aligned}
\theta^T &= [\theta_1' \dots \theta_T']' \\
H^T &= \begin{bmatrix} h_{11} & h_{21} & \cdots & h_{n1} \\ h_{12} & h_{22} & \cdots & h_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ h_{1T} & h_{2T} & \cdots & h_{nT} \end{bmatrix} \\
y^T &= [y_1' \dots y_T']'
\end{aligned}$$

Finally, let $\beta = [\beta_{21} \quad \beta_{31} \quad \cdots \quad \beta_{nn-1}]$ and $\sigma = [\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_n]$.
The posterior density $p(\theta^T, Q, \sigma, \beta, H^T | y^T)$ summarizes the beliefs about the free parameters in the model. To simulate the posterior we executed 350,000 draws of a Metropolis-within-Gibbs sampler, and discarded the first 300,000 to allow for convergence. A detailed description of algorithm used can be found in Navarro (2012) or in Cogley and Sargent (2005). Also, in the Appendix we describe the priors used. We checked convergence by inspecting recursive mean plots of various parameters and by comparing results across parallel chains starting from different initial conditions. All results presented correspond to estimated means.

2.2.2 Emprical Results

Our observed variable is $y_t = [\Delta Y_t \quad \pi_t \quad T - Bill_t \quad CS_t]'$, a 4×1 vector including GDP growth, CPI inflation, 3 months T-Bill rate and average spread as computed by Gilchrist and Zakrajsek (2012). Data frequency is quarterly and covers the period from 1973:1 to 2007:4[6].

Figure 4 shows the evolution of the innovation variances (diagonal elemenst of R_t). The picture display considerable time variation in innovations variance, evidence in favor of a

[6]We discarded the years of financial crises in order to focus in more “normal times”.

time-varying model. In particular, and in line with the results found by Cogley and Sargent (2005), innovations for the T-Bill equation spike up around 1980 and rapidily decrease after that. Similarly, both innovations for GDP growth and inflation reach a peak around 1979 and start decreasing from thereafter. Finally, higher innovations in the credit spreads equations come earlier, around 1975, and more remarkably around 2001. It is also interesting to note that, while the other variables experienced a clear decrease in varaince innovations during the 80's, this is not the case for credit spreads which shows a small but steady increase in it's variance. Also, note how credit spreads volatility increases at the end of the sample: something particulary interesing since we are not using observations from the years of the financial crises. In the same line, note that only innovations to the spread equation increase during the 2001 recession.

Next we describe beliefs about long run behavior of these variables. In particular, we rewrite (1) in companion form as follows

$$z_t = c_t + F_t z_{t-1} + \nu_t$$

where c_t and F_t are functions of θ_t and ν_t is a function of R_t and ξ_t, Then, define long-run mean μ_t and variance Σ_t as[7]

$$\begin{aligned} \mu_t &= (I - F_t)^{-1} c_t \\ \Sigma_t &= F_t \Sigma_t F_t' + \Xi_t \end{aligned}$$

where $\Xi_t = \mathbb{E}(\nu_t \nu_t')$. Figures 5 shows the results for the mean and 6 show the results for the variance. Regarding means we observe a decrease in expected inflation starting in the early 80's contemporaneously with an increase in GDP growth expectation that settles around

[7] Note that these long run moments are well defined as long as all the eigenvalues of F_t are inside the unit circle. We checked that this holds for every period.

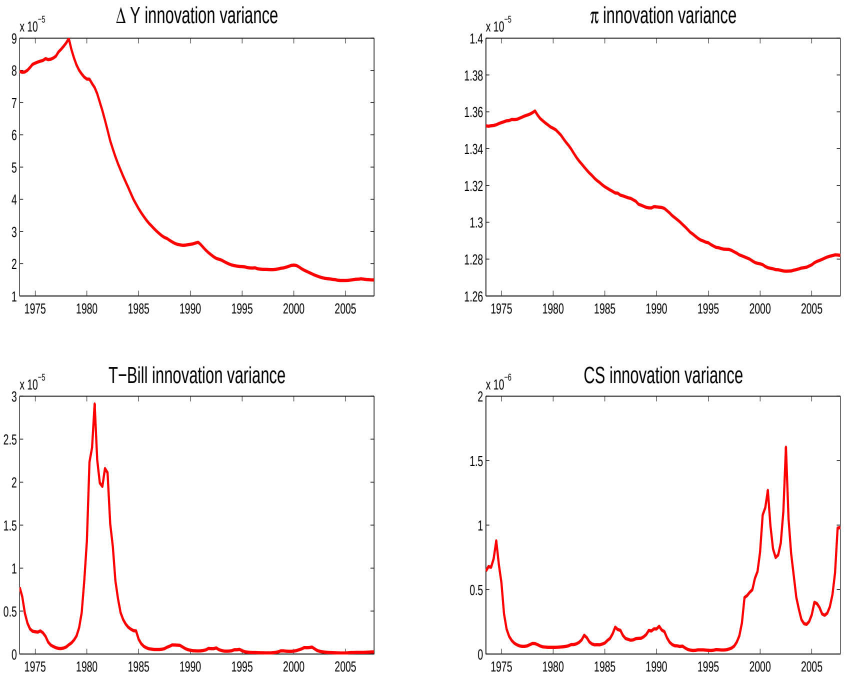

Figure 4: diagonal elements of R_t

3.5% in the early 90's. Also, an increase in T-Bill expectations in the early 80's, from 4.5% in 1980 to 6.5% in 1985 and back to 4.5% in 1990. Finally, although non-monotonically, we observe an increase in the long-run beliefs of credit spreads moving from below 1% at the beginning of the sample towards a peak of 1.5% by the end of the 90's. Note that this is not the realization of credit spreads, but the long-run belief about it. Regarding variances, we observe a similar pattern to that found in innovation variances (Figure 4): only credit spreads seems to have increased its volatility during the last couple decades. All together, we find a large moderation in GDP growth, inflation and T-Bill, but a slowly and steady increase in both the variance and the mean of credit spreads. In other words, while we find evidence of the "Great Moderation" for the US economy, this results does not extend to credit spreads, speacially in their long run values.

We conclude this section with two messages in mind: First, there is strong statistical

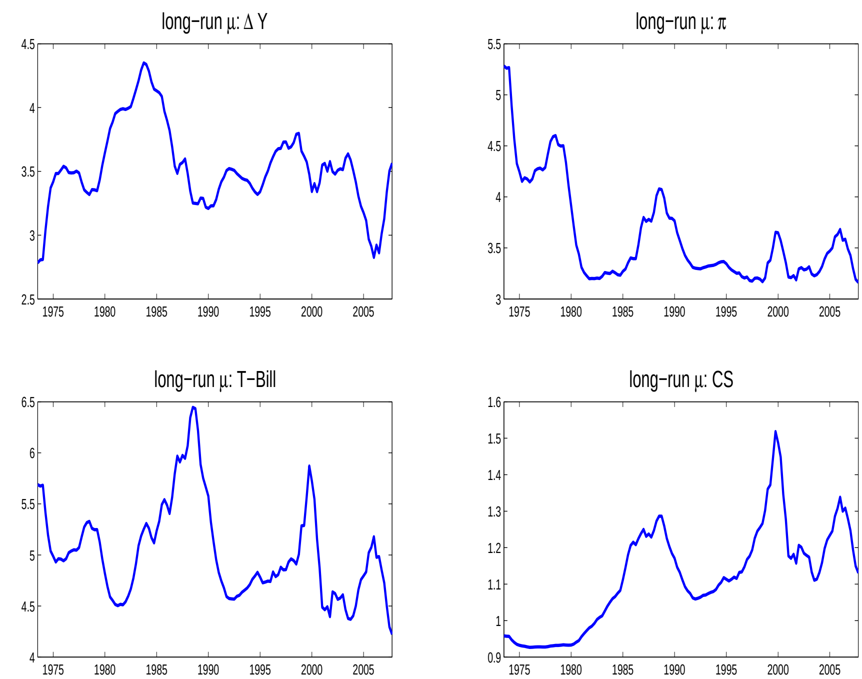

Figure 5: long-run mean μ_t

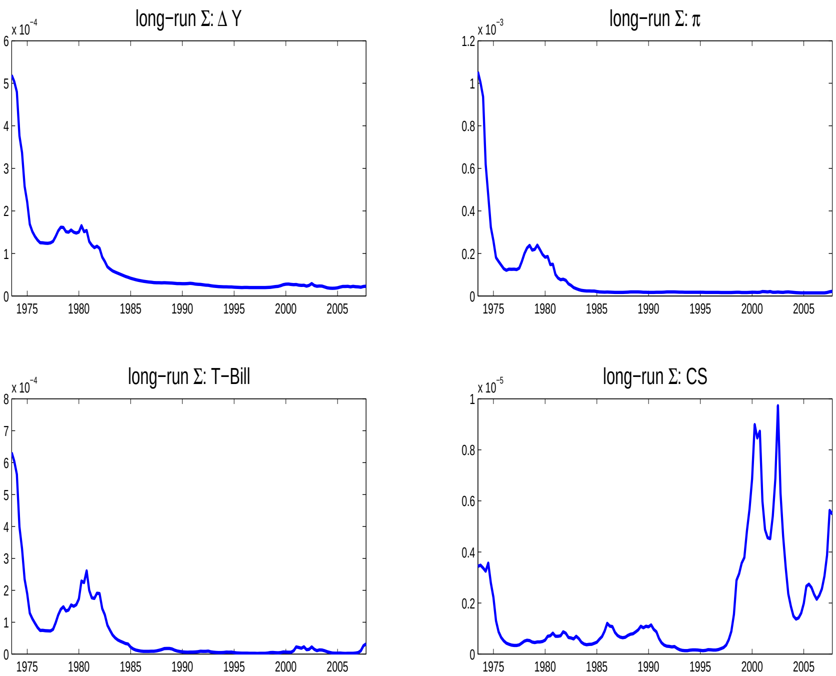

Figure 6: long-run variance Σ_t

evidence of comovements between credit spreads and aggregate variables. Furthemore, relations seems not to be linear. Second, there was an interesting change in the behaviour of credit spreads during the last years, in particular one that is not in line with the "Great Moderation". We understand this as enough motivation to develope a model of default that can help understanding some these patterns, this is our aim in next section.

3 Model

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. The economy is populated by three type of agents: workers, entrepreneurs and investors. Each entrepreneur runs her own firm: every period they can hire labor, buy capital for next period and issue debt. Workers are assumed to be hand-to-mouth, and for further simplicity we assume that they supply labor inleastically[8]. Investors buy and sell all the debt issued by entrepreneurs, but they can not default. There are two limited market participation assumptions: first, only entrepreneurs can produce output, and second, investors are the only ones with the skills to insure entrepreneurs against idiosyncratic risk[9].

Entrepreneurs productivity has to components: an aggregate and an idiosyncratic one. Both evolve stochastically, and while the aggregate may have some persistance we assume that the idiosyncratic is *i.i.d.* across time and agents. Every period, before picking their new portfolio but after production took place, entrepreneurs can default on their debt. The cost to do so is loosing a fraction of their capital returns, but there is no other punishment. In particular, they can access credit markets again that very same period without any further complications. This simplifying assumption is the core of the tractability achieved in this model.

[8]Adding a static consumption-leisure decision can easily be done. See Bigio (2011) for an example.

[9]Having an industry equilibrium model is extremelly desirable and is top among our future reasearch priorities. See Cooley and Quadrini (2001) for an example.

We describe entrepreneurs and investors decisions next.

3.1 Entrepreneurs

Entrepreneur's problem can be divided in three sub-problems: the production decision, default decision and portfolio decision. We show each one in turn.

Production Decision

Entrepreneur's have access to a constant returns to scale technology given by

$$y = (axk)^{1-\alpha} n^{\alpha} \tag{5}$$

where a is idiosyncratic productivity, x is aggregate productivity, k is capital and n is labor demand. Since at the beginning of the period capital is given, entrepreneur's production decision accounts for choosing labor demand in order to maximize current profits only. Let s denote the aggregate state of the economy, which includes x and more elements to be detailed later. Then, his problem is

$$\pi(a, k, s) = \max_{n} \left\{ (axk)^{1-\alpha} n^{\alpha} - w(s)n \right\}$$

where $w(s)$ are wages when the state of the economy is s. The following proposition characterizes entrepreneur's optimal policies regarding production.

Lemma 1 (Optimal Policies Regarding Production) *For an entrepreneur with state*

(a, k, s) *optimal policies are given by*

$$n(a,k,s) = \left(\frac{\alpha}{w(s)}\right)^{\frac{1}{1-\alpha}} axk \tag{6}$$

$$y(a,k,s) = \left(\frac{\alpha}{w(s)}\right)^{\frac{\alpha}{1-\alpha}} axk \tag{7}$$

$$\pi(a,k,s) = \psi(s)ak \tag{8}$$

$$\text{where} \tag{9}$$

$$\psi(s) = xw(s)^{-\frac{\alpha}{1-\alpha}}\left[\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right] \tag{10}$$

(All proofs are left for the Appendix) The results are intuitive: policies for labor demand and output produced are increasing in productivity and decreasing in the price of the input. Also, it worth noting that profits are linear in capital. In particular, return per unit of capital invested is independent of the level of capital, which is due to the constant reurn to scale assumption and will allow us to obtain simple solutions later.

Default Decision

We assume that, upon default, an entrepreneur looses a fraction $\varphi \in (0, 1)$ of his capital. In this case, he would also loose the returns on that capital. Let b denote entrepreneur's savings, then an entrepreneur with state (a, b, k, s) defaults if

$$b + [(1-\delta) + \psi(s)a]k < (1-\varphi)[(1-\delta) + \psi(s)a]k \tag{11}$$

The left hand side in equation (11) is wealth if debt is repayed and the right hand side is wealth upon default, where we already used the expression for the firm profits as in Lemma 1. This inequality simply states that default decision is done in order to maximize current wealth, which naturally derives from our assumptions regarding default punishment.

Equation (11) can be rewritten as

$$a < -\frac{1}{\psi(s)}\left[(1-\delta)+\frac{\chi}{\varphi}\right] \tag{12}$$

where $\chi = b/k$ is savings over capital (negative of leverage). Then, there is a treshold $\underline{a}(\chi, s)$ below which entrepreneurs default

$$\underline{a}(\chi, s) = -\frac{1}{\psi(s)}\left[(1-\delta)+\frac{\chi}{\varphi}\right] \tag{13}$$

which depends on both the aggregate state s and the porfolio decision χ. Two comments are pertinent: first, χ was chosen last period and with this also the probability of default this period; second, an entrepreneur with higher leverage (lower χ) has a larger default treshold and consequently higher probability of default. Thus, this simple assumption captures the two basic intuitions in most of the literature: higher leverage (more exposure) implies higher risk, and more productive agents default less. Furthermore, default rate is lower in states s where profits per unit of capital $\psi(s)$ are higher. We will show later that this coincides with good moments of the economy, which is consistent with data.

Finally note that, other than the aggregate state s, all the risk involved in an entrepreneurs portfolio decision is captured by χ. This is because the idiosyncratic shock a is *i.i.d.* accross agents and over time and thus previous information about productivity does not help predicting future default probabilities. In other words, the price of a bond issued by an entrepreneur must depend only on the aggregate state and his leverage decision. Thus, let $p(\chi', s)$ be the price of a bond from an entrepreneur picking a portfolio with leverage for next period $\chi' = b'/k'$ when the aggregate state is s. Figure 7 shows how the price function will look in equilibrium: for lower values of χ the default probability is larger and the value of the debt lower, as χ increases also does the price and eventually the bond becomes risk-free (no possible realization induces default) and the price remains flat.

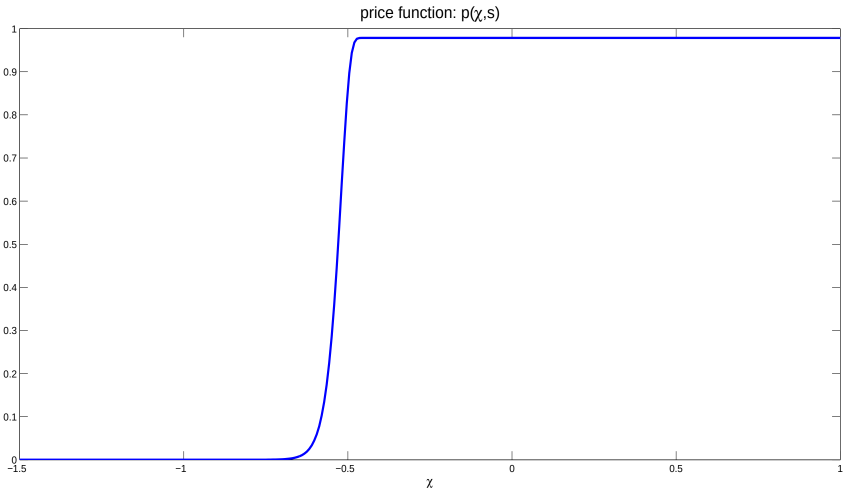

Figure 7: Price function $p(\chi, s)$

Also, note that the model endogenously generates a continuum of assets which risk is indexed by χ: each entrepreneur must pick one these assets to fund himself and obtain the corresponding price. Thus, even with this simplifying assumptions, the asset structure of this economy is very rich.

Entrepreneur's Portfolio Decision

Finally, we turn to the portfolio decision. Let ω be the *ex-post* default wealth held by the entrepreneur and let $v^e(\omega, s)$ be his value when the aggregate state is s. Then, his problem is

$$
\begin{aligned}
v^e(\omega, s) &= \max_{\{c,b',k'\}} \{\log(c) + \theta\beta\mathbb{E}[v^e(\omega', s')|s]\} \qquad (14)\\
c + p(\chi', s)b' + k' &\leq \omega \\
\chi' &= \frac{b'}{k'} \\
\omega' &= \begin{cases} (1-\varphi)[(1-\delta) + \psi(s')a']k' & \text{if } a' < \underline{a}(\chi', s') \\ b' + [(1-\delta) + \psi(s')a']k' & \text{if } a' \geq \underline{a}(\chi', s') \end{cases} \\
c \geq 0, \quad k' &\geq 0, \quad s' = \Gamma(s) \qquad (15)
\end{aligned}
$$

The problem is resonable standard. Entrepreneurs chooses consumption, debt and capital in order to maximize utility today plus his continuation value. He must satisfy his budget constraint but at the same time internalizes the fact that different portfolio decisions χ' affect the price of his debt. Also, the law of motion for his wealth follows from the default decision detailed above. Finally he faces some non-negativity constraints and understands the law of motion for the aggregate state (i.e: has rational expectations).

Note that we do not need to know as individual states the entrepreneur's default history neither how he obtained his wealth. Again, this is because of the default punishment assumption: having obtined this wealth by defaulting or not is irrelevant, he has access to the same financial instruments. Also, we are denoting entrepreneur's discount factor as $\theta\beta$, where β will be the discount factor of the investor. We will assume $\theta \in (0, 1)$ so that this a lower discount factor for the entrepreneur prevents him to save out of the financial friction. Denoting it as the product of two numbers is for algebraical convenience.

The following proposition charactreizes entrepreneur's optimal portfolio and consumption decision

Proposition 1 (Entrepreneur's Optimal Portfolio and Consumption Decision) *Optimal portfolio and consumption policies for the entrepreneur are*

$$c(\omega, s) = \theta\beta\omega \tag{16}$$

$$b'(\omega, s) = \frac{\chi'(s)}{p(\chi'(s), s)\chi'(s) + 1}\theta\beta\omega \tag{17}$$

$$k'(\omega, s) = \frac{1)}{p(\chi'(s), s)\chi'(s) + 1}\theta\beta\omega \tag{18}$$

where

$$\chi'(s) \in \arg\max_{\chi'} \mathbb{E}\Bigg\{ \int_0^{\underline{a}(\chi',s')} \log\left(\frac{(1-\varphi)[(1-\delta)+\psi(s')a]}{p(\chi',s)\chi'+1}\right) f(a)da \qquad (19)$$
$$+ \int_{\underline{a}(\chi',s')}^{\infty} \log\left(\frac{\chi'+(1-\delta)+\psi(s')a}{p(\chi',s)\chi'+1}\right) f(a)da \mid s \Bigg\}$$

and f is the probability density function of a.

The results are in some extent standard. The form of the consumption policy is due to the logarithmic utility assumption[10]. The expressions for savings and capital comes from the definition of χ and the budget constraint using the optimal policy for consumption. Equation (19) is the new interesting term and where most of the action in this model is concentrated: The term $p(\chi', s)\chi' + 1$ is the cost of the portfolio choice per unit of capital, and thus the integral is just the expected return (i.e.: payoff over cost) next period per unit of capital taking into account the default decision. Also, we have the log of the return because of the closed form solution of this problem (see the Appendix).

Note that equation (19) includes aggregate but no individual states. Thus, entrepreneurs make different leverage decisions over time but, in any given period, they make the same leverage decision. Consequently, they are all issuing equally risky debt and funding their firms with the same cost (per unit of capital). This result is what makes the model simple and tractable: for understanding risk in this economy we do not need to keep track of the distribution of portfolio decisions since they all choose the same. However, note that wealthier entrepreneurs will run larger firms by issuing more debt and buying more capital: they leverage up more wealth.

It can be seen from these results that the distribution of idiosyncratic characteristics, like

[10]More generally, linear policies arises as a combination of of homogenous preferences together with feasible sets homogenous of degree one. The logarithmic case is particulary more simple, but linearity would still hold with, for instance, CRRA preferences with different risk aversion parameters. See Alvarez and Stokey (1998).

wealth, productivity or capital, will not be a state for the economy. Since policies are linear and every portfolio is, modulo a constant, identical, we will be able to reduce significantly the dimensionality of the state space. Also, equation (19) is the only step that can not be solved by hand: by knowledge of the optimal leverage decision $\chi'(s)$ we can compute any other quantity. This makes the model quantitatively attractive.

Next we turn to investors, the ones who actually buy and sell entrepreneur's debt.

3.2 Investors

Every period, investors must pick consumption and a portfolio using all the different type of bonds available χ's. The advantage they have is that, by pooling debt of many different entrepreneurs, they "wash out" the idiosyncratic risk. In particular, their return on lending is the rate of non-defaulting entrepreneurs.

Let ω be the *ex-post* default wealth that the investor holds. Then, his problem is as follows

$$\begin{aligned} v^i(\omega, s) &= \max_{c,\{b'(\chi)\}_\chi} \left\{\log(c) + \beta\mathbb{E}[v^i(\omega', s')|s]\right\} \qquad (20) \\ c + \int p(\chi, s)b'(\chi)d\chi &\leq \omega \\ \omega' &= \int R(\chi, b'(\chi), s')b'(\chi)d\chi \end{aligned}$$

where $R(\chi, b'(\chi), s') = 1 - \mathbf{1}\{b'(\chi) > 0\}F(\underline{a}(\chi, s'))$ is the rate of non-defaulted bonds. The return depends on the portfolio decision $b'(\chi)$ because we assumed that investor can not default: if $b'(\chi) < 0$ he will pay one unit next period, while if $b'(\chi) > 0$ he will receive as return the non-defaulted bonds. Then, investor's problem is a standard (dynamic) portfolio decision problem: he has to choose today's consumption as well as a portfolio in order to maximize current utility plus continuation value. Note, as mentioned before, that the discount factor is larger for investor that for the entrepreneur, this will induce lending from former to the latter[11].

Next proposition describes investors optimal portfolio and consumption decision.

Proposition 2 (Investor's Optimal Portfolio and Consumption Decision) *Optimal policies for the investor are*

$$\begin{aligned} c(\omega, s) &= (1-\beta)\omega & (21) \\ p(\chi, s) b'(\chi, \omega, s) &= \gamma(\chi)\beta\omega & (22) \end{aligned}$$

where the portfolio weights $\gamma(\chi)$ *are such that*

$$\begin{aligned} \{\gamma(\chi)\} &= \arg\max_{\{\gamma(\chi)\}} \mathbb{E}\left[\log\left(\int \frac{R(\chi, b'(\chi), s')}{p(\chi, s)}\gamma(\chi)d\chi\right)\right] & (23) \\ s.\ to & \quad \int \gamma(\chi)d\chi = 1 \end{aligned}$$

Also,

$$p(\chi, s) = \mathbb{E}\left[\Lambda(s, s')R(\chi, b'(\chi), s')\right] \quad \forall\chi \text{ and } b'(\chi) \neq 0 \qquad (24)$$

where $\Lambda(s, s') = \beta c(\omega, s)/c(\omega', s')$ *is the stochastic discount factor of the investor and wealth next period is determined by the portfolio decision as* $\omega' = \int R(\chi, b'(\chi), s')\gamma(\chi)\frac{\omega}{p(\chi,s)}d\chi$

The proposition is rather intuitive. As discussed above, linear policy for consumption comes from the logarithmic utility assumption. The portfolio is choosed in order to maximize the expected return per unit of wealth invested, and the logarithmic of the return comes from the closed form solution for the value function (see Appendix). Finally, equation (24) is

[11] There is also a problem to define a steady state in this economy without this assumption. In particular, if $\theta = 1$ investors disappear in an enviroment without aggregate uncertainty. The reason is simply that entrepreneurs would still suffer from the idiosyncratic shock and will desire to save, a precautionary saving motive; investors not facing any risk will borrow from the formers and slowly fade out their wealth. In the limit, only entrepreneurs would survive.

a standard pricing equation, but we must keep this caveat because of the discontinuity at $b'(\chi) = 0$. This last equation is what will let us price every asset in the economy, even those that are not traded in equilibrium.

3.3 Equilibrium

We formally define an equilibrium for this economy next.

Definition 1 *A recursive competitive equilibrium for this economy is given by: price functions* $\{p(\chi', s), w(s)\}$*, value functions* $\{v^e(\omega, s), v^i(\omega, s)\}$*, investor's policies* $\{c^i(\omega, s), b^{i\prime}(\chi, \omega, s)\}$*, and entrepreneur's policies* $\{n(k, a, s), c^e(\omega, s), k^{e\prime}(\omega, s), b^{e\prime}(\omega, s)\}$ *such that, given prices:*

(i) $\{n(k, a, s), c^e(\omega, s), k^{e\prime}(\omega, s), b^{e\prime}(\omega, s)\}$ *are optimal for the entrepreneur*

(ii) $\{c^i(\omega, s), b^{i\prime}(\chi, \omega, s)\}$ *are optimal for the investor*

(iii) markets clear:

- *(labor market)* $\int n(b, a, s, k) = \bar{n}$
- *(bonds market)* $\int b^{e\prime}(\omega, s) + \int b^{i\prime}(\chi, \omega, s) = 0 \quad \forall \chi$

(iv) law of motion $s' = \Gamma(s)$ *and a transition probability* $P(s, s')$

where $\bar{n}$ *is hours supplied by workers every period.*

There is a slightly abuse of notation in the bond's market clearing condition: when integrating over entrepreneurs, we mean only those who issue that type of debt χ.

So far we avoided an explicit characterization of the aggregate state. In general, the entire distribution of portfolios, capital and wealth, as well as the aggregate productivity, would be part of the state. However, as we show next, the aggregate state is simply total capital, total debt and aggregate productivity. We turn next to an equilibrium characterization to better explain this.

3.3.1 Equilibrium characterization

The following proposition characterizes the equilibrium and law of motion for aggregate variables

Proposition 3 (Equilibrium Characterization) *Let K and B^i be the aggregate capital and aggregate savings entrepreneurs and investors start this period with respectively. The aggregate state of the economy is $s = (x, K, B^i)$. Furthemore, the evolution of these variables are as follows*

$$B^{i\prime}(s) = -\chi'(s)\beta[W^i(s) + \theta W^e(s)] \tag{25}$$

$$K'(s) = \beta[W^i(s) + \theta W^e(s)] \tag{26}$$

where $\chi'(s)$ is given by (19) and $W^e(s)$ and $W^i(s)$ are ex-post default wealth held by entrepreneurs and investors respectivelly when the aggregate state is s. These are given by

$$W^i(s) = B^i[1 - F(\underline{a}(\chi, s))] \tag{27}$$

$$\begin{aligned} W^e(s) &= [1 - \varphi F(\underline{a}(\chi, s))](1-\delta)K + [1 - F(\underline{a}(\chi, s))]B^e \\ &+ (1-\varphi)\psi(s)K \int_0^{\underline{a}(\chi,s)} af(a)da \\ &+ \psi(s)K \int_{\underline{a}(\chi,s)}^{\infty} af(a)da \end{aligned} \tag{28}$$

where $\chi = -B^i/K$ is the leverage level the economy started with this period and

$$\psi(s) = (1-\alpha)x^{1-\alpha}\left[\mathbb{E}(a)K\right]^{-\alpha}\bar{n}^{\alpha} \tag{29}$$

Finally, price of the debt traded is

$$p(\chi'(s), s) = -\frac{1}{\chi'(s)}\frac{W^i(s)}{W^i(s) + \theta W^e(s)} \tag{30}$$

Two comments are pertinent. First, both investor savings and entrepreneur's capital are increasing in aggregate wealth of the economy (as long as $\chi'(s) < 0$, which is always the case). This is an intuitive result: and economy with more wealth accumulates more capital. Second, the price of the bond actually traded depends on wealth distribution: in states where investors hold a larger fraction of aggregate wealth, they demand more bonds and the price of those increases as can be seen in equation (30). Thus, in this economy wealth distribution matters for asset prices.

The form from the law of motion for capital and debt as in equations (25) and (26), as well as the expression for the price in (30), comes from aggregating individual policies and imposing market clearing. This aggregation is particulary simple because of the linearity in individual policies. Notice that for computing elements in this proposition, as well as the ones in (19), we only need knowledge of (x, K, B^i). This is then our state variable. Thus proposition completely characterizes dynamics for the aggregate variables in the economy, please see the Appendix for a complete derivation and the details about the computational algorithm to solve the model.

4 Example: A One Period Model

In this section we develope a strip-down version of our model to explain the main mechanism. In particular, we want to show how the price of the debt changes with the entrepreneur's portfolio decision. Furthermore, how he internalizes this when making his funding decision in order to maximize the trade-off between borrowing cost and expected return.

Assume a risk-neutral entrepreneur who starts in period 0 with wealth ω_0 and can pick between defaultable debt and capital. The later's pay-off is a random variable $a \sim F$ that comes in period 1 and capital fully depreciates. Thus, investing k units of capital delivers ak units in period 1. Same as before, upon default he looses a fraction φ of his capital return, and thus the default treshold is $\underline{a}(\chi) = -\frac{\chi}{\varphi}$. Assume also a risk-neutral lender that buys debt at a price equal to the repayment probability. Then, the problem of the entrepreneur is

$$\begin{aligned}
\max_{\{b,k\}} \quad & \mathbb{E}(\omega_1) \\
\text{s. to} \quad & \\
p(\chi)b \quad + \quad & k \leq \omega_0, \quad \chi = \frac{b}{k} \\
\omega_1 \quad = \quad & \begin{cases} (1-\varphi)ak & \text{if } a < \underline{a}(\chi) \\ b + ak & \text{if } a \geq \underline{a}(\chi) \end{cases}
\end{aligned}$$

It can easily be seen that the problem can be transformed in chooshing the leverage ratio χ only[12]. Then, the first order condition for χ reads as follows

$$\int_0^{\underline{a}(\chi)} \frac{(1-\varphi)a}{p(\chi)\chi+1} f(a)da + \int_{\underline{a}(\chi)}^{\infty} \frac{\chi+a}{p(\chi)\chi+1} f(a)da = \frac{1-F(\underline{a}(\chi))}{p'(\chi)\chi+p(\chi)} \tag{31}$$

where $f(a) = F'(a)$ is the density.

Lets think of the marginal cost and benefit per unit of capital of a leverage decision χ. The value paid (per unit of capital) of a portfolio with leverage χ is $p(\chi)\chi+1$, and the pay-off is $(1-\varphi)a$ in the default region and $\chi + a$ otherwise[13]. Thus, the left hand side of (31) is the marginal return (i.e.: pay-off over investment) of the portfolio per unit of capital. The cost of funding generally is the interest rate, in this case $1/p(\chi)$. However, two more things come

[12]Just in case: using the budget constraint and definition of χ, we have that next period wealth is

$$\omega_1 = \begin{cases} \frac{1}{p(\chi)\chi+1}(1-\varphi)a\omega_0 & \text{if } a < \underline{a}(\chi) \\ \frac{1}{p(\chi)\chi+1}[\chi+a]\omega_0 & \text{if } a \geq \underline{a}(\chi) \end{cases}$$

[13]Since, in this region, the entrepreneur repays its debt which is χ per unit of capital.

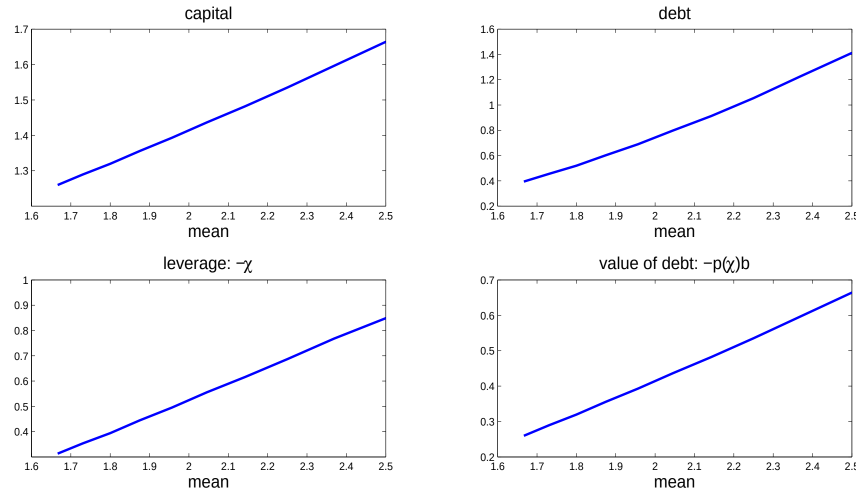

Figure 8: One Period Model: Optimal Policies

into play now. One one hand, debt is not repayed in every state but only with probability $1 - F(\underline{a}(\chi))$, which in principle lowers the cost of funding[14]. On the other hand, portfolios with different default probabilities carry different debt prices: in other words, $p'(\chi) \neq 0$. In particular, a more risky debt comes with a lower price increasing the cost of funding[15]. This trade-off is what we find in the right hand side of (31). As usual, in the optimum the entrepreneur equals marginal benefit and marginal cost, but in this case he does it by understanding that he can aslo affect the cost by offering debt with different risk levels.

Figure 8 shows the optimal solutions for different expected return values[16]. As we can see, for higher expected productivities (i.e.: higher $\mathbb{E}(a)$) the entrepreneur borrows more and buys more capital, this is done with a value of debt $(-p(\chi)b)$ that increases as well. However, notice that he leverages up more which implies a more risky decision. We will show next section that this is not any longer the case whith concave preferences.

This examples contains most of the intuition in our model. The entrepreneur makes

[14]Ceteris paribus, the entrepreneur pays less often.

[15]Note that, $\frac{\partial(p(\chi)\chi+1)}{\partial\chi} = p'(\chi)\chi + p(\chi)$ is the marginal change in the cost of the funds.

[16]We solve the example assuming an exponential distribution $f(a) = \lambda \exp^{-\lambda a}$ with $\lambda \in [0.4, 0.6]$.

an optimal portfolio decision by considering marginal return of his project as well as by internalizing the fact that he can affect the cost of funds. On top of this, periods of expected higher return (higher means) come with more capital accumulation and higher borrowing. We show next that most of this intuition remains in the more complex model.

5 Quantitative Analysis

We now turn to a quantitative assesment of the model.

5.1 Calibration

The model is calibrated so that one period represents a quarter. Some of the parameters in the model are standard and we follow typical calibrations. For the investor discount factor we pick $\beta = 0.99$ which implies a risk-free of 1% in this economy with no aggregate shocks[17]. For the capital depreciation we picked $\delta = 0.025$ and $\alpha = 0.64$ for the share of labor in output which is standard. We also use $\bar{n} = 1/3$ which represents an eight hours day of work. Finally, we assume that aggregate productivity follows an $AR(1)$ process in logs: $\ln(x_t) = \rho_x \ln(x_{t-1}) + \sigma_x \varepsilon_t$, where we used $\rho_x = 0.9$ and $\sigma_x = 0.007$ which is close to typically used values.

The model non-standard parameters are φ, θ and the distribution of the idiosyncratic shock. For the former we choosed $\varphi = 0.45$ which imply an average leverage of 0.46, close to the observed in data. For the idiosyncratic shock we assumed a log-normal distribution $\ln(a) \sim N(\mu_a, \sigma_a^2)$ with paramteres such that $\mathbb{E}(a) = 1$ and $Var(a) = 0.125$[18]. Finally, we choose $\theta = 0.7$ to match an average default rate of 2.5%, which is close to data[19]. A

[17]Note that the average risk-free rate of the simulations needs not to be this number since the economy simulated has aggregate uncertainty.

[18]At the moment we are not matching the mean of spreads, but we could do it with this parameters.

[19]We used the time series of deliquency rate, which average is 3% for the available period of 1987:1 to 2011:1. Note that delinquency rate is payments delayed for at least two months, although some can be

Parameter	Value
β	0.99
δ	0.025
α	0.64
$\bar{n}$	1/3
ρ_x	0.9
σ_x	0.007
φ	0.45
θ	0.7
$\mathbb{E}(a)$	1
$Var(a)$	0.125

Table 2: Calibration

similar value is used by Arellano, Bai, and Kehoe (2011) which shows how important higher moments become when using global solutions. Table 2 shows the parameters value.

5.2 Policies and Equilibrium

Figure 9 shows the most important policies of the model: capital ($K'(s)$), debt ($-B^e(s)$), leverage ($-\chi'(s)$) and value of issued debt ($-p(\chi', s)B^e(s)$). All of them are plot as a function of the aggregate productivity x[20]. As we can see, as productivity becomes higher, entrepreneurs accumulate more capital and also issue more debt to finance this investment. Since both increase, is not clear what would happen with the ratio $\chi' = B^{e\prime}/K'$. Nevertheless, we can see that leverage decreasaes which means that this expansion is done in a less risky fashion. In the same line, the total funds obtained by entrepreneurs increase.

Another way to see how risk decreases in good moments is by looking at spreads in the economy. Remember that, even if there is only one type of bond traded, we can price any of the bonds type χ in the economy. In particular, if $\chi > 0$ there is no risk involved in that bond (the entrepreneur is actually saving) and we can consider that as a risk-free bond.

repayed later. This is why we target something slightly below the average. The data was obtained from Board of Governors.

[20]The other elements of the state (investors savings and aggregate capital) are held to intermediate values.

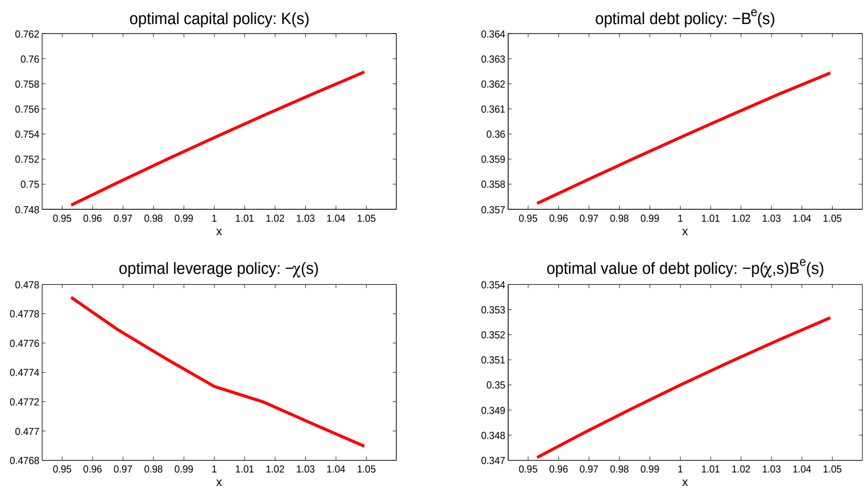

Figure 9: Policies as function of x

Note as well that there are many risk-free bonds in this economy (for instance, any $\chi > 0$) but in equilibrium they all have the same price, as we can see in Figure 7. Thus, let $p^f(s)$ be the price of the risk-free bond when the aggregate state is s and define spreads in this economy as

$$spread = \frac{1}{p(\chi'(s), s)} - \frac{1}{p^f(s)}$$

where $\chi'(s)$ is entrepreneur's leverage choice when the aggregate state is s as given in equation (19). Figure 10 shows the results as a function of aggregate productivity x: in same line as before, periods of higher productivity are associated with lower spreads.

The discussion in this section highlights two opposing effects. On one hand, when the economy is in an expansion, capital is more profitable and entrepreneurs want to borrow more in order to increase investment. Thus, spreads can potentially be higher because of the increase in borrowing. On the other hand, during these periods default probabilities are smaller just because the aggregate productivity is high. The results in this section indicate that the second force is stronger: the economy can jointly deliver more and less risky investment during expansions which, as we showed in the empirical part, is in line with

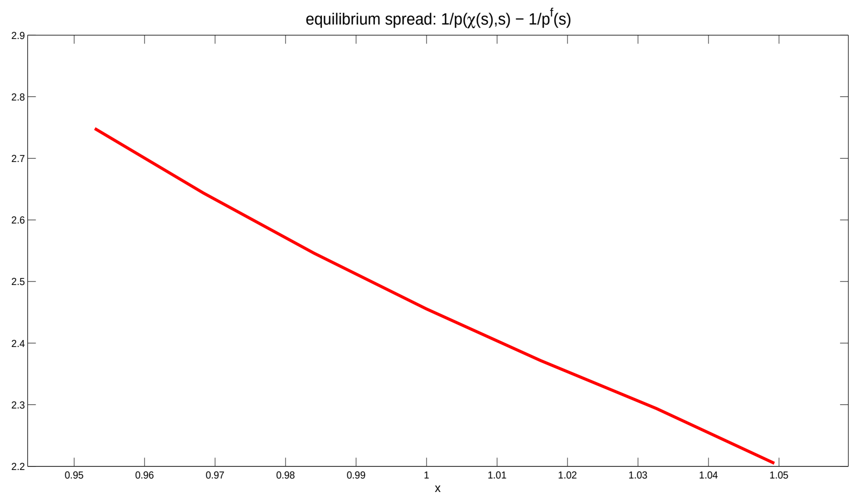

Figure 10: Spread as function of x

the data.

5.3 Simulation

In this section we simulate the economy and compare some moments with their empirical counterparts. Table 3 shows results for some of the most important variables in our model. In same line with data, the model generates an output growth more volatile than consumption but less than investment. Also, credit spreads are less volatile than output although the difference with data is large. The model can also deliver high average spreads, even higher than observed ones. Regarding correlations with output growth, the model predicts the right sign for all of the variables. Consumption and investment are positively correlated, while credit spreads, default rate and market leverage are negatively correlated. Furthermore, for a one shock economy, results look very satisfactory.

Figure 11 is the same scatter plot we showed in the empirical part for the simulated economy. The model predicts very well the relations qualitatively and, as just discussed, is quantitatively close. In general lines, the model has a good empricial fit.

	Model	Data
spread	3	1.67
$\sigma(\Delta C)/\sigma(\Delta Y)$	90%	75%
$\sigma(\Delta I)/\sigma(\Delta Y)$	154%	118%
$\sigma(spread)/\sigma(\Delta Y)$	1%	41%
Default Rate	2.6%	2.5%
Leverage	0.46	0.45
corr with ΔY		
ΔC	0.60	0.96
ΔI	0.66	0.88
Spread	-0.56	-0.38
Default Rate	-0.46	-0.33
Market Leverage	-0.51	-0.11

Table 3: Moments

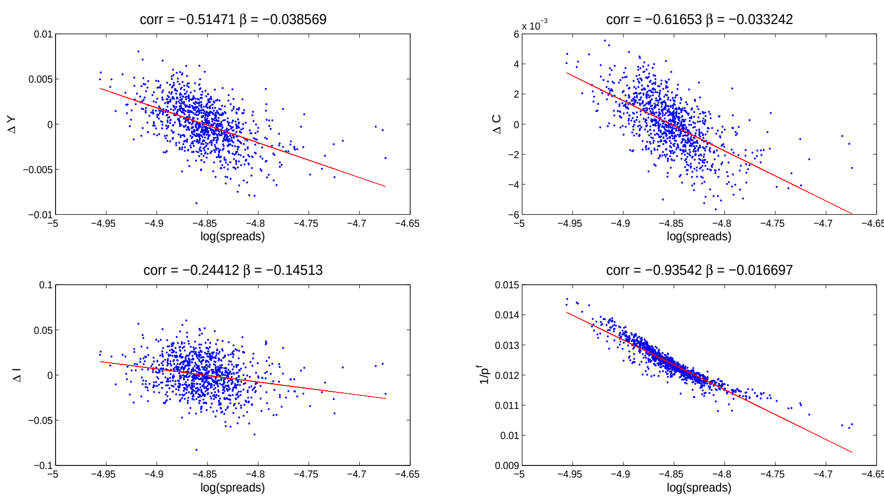

Figure 11: Simulation crossplot

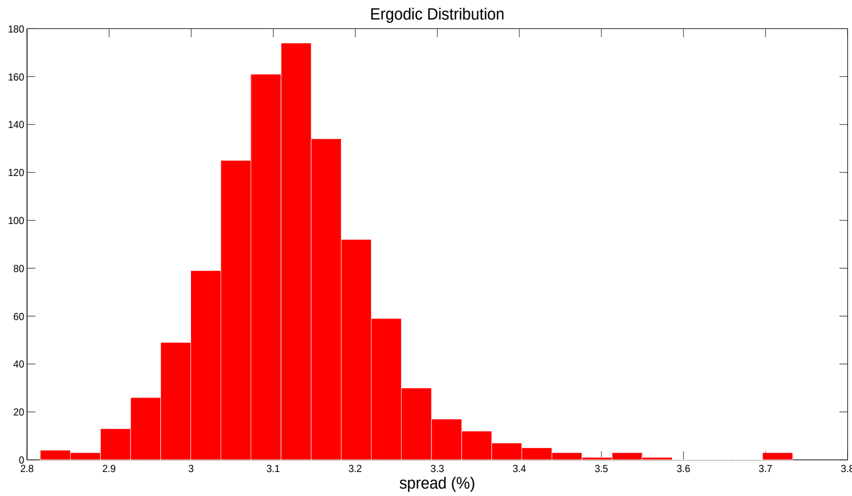

Figure 12: Simulation Spreads Distribution

5.4 Asymmetries

One of the contributions of this paper is to develope a model that accepts a (tractable) global solution. The advantage of this is that every moment is present and allow us to explore the potential non-linearities. We do this in this section.

Table 4 computes the skewness and kurtosis, the two most standard measures of asymmetry, for the variables showed in the emprical part. As remarked before, the data is far away from symmetric: skewness is statistically different from zero and the kurtosis is different from three. Thus, data is in support of non-linear solutions.

Most of our focus has been in spreads, and we can verify that the model correctly predicts skeweness and kurtosis far from symetry. Furthermore, the skewness in the model is in the same direction that data: it has a larger right tail. In other words, the model can predict the observed ocasionally jump to high spreads. Figure 12 shows the ergodic distribution of spreads in the model. Admittedly, the asymmetry in the model is not present in the other variables, but undoubtedly this is a first step in the right direction[21].

[21] A similar argument for the presence of non-linearities can be found in Brunnermeier and Sannikov (2012).

	Skewness			
	spread	ΔY	ΔI	ΔC
Data	1.55	−0.21	−0.13	−1.11
Model	0.72	0.003	0.03	0.009
	Kurtosis			
	spread	ΔY	ΔI	ΔC
Data	5.26	5.25	4.63	6.71
Model	6.91	2.98	3.05	3.04

Table 4: Skewness and Kurtosis

Another way to look at the asymmetris is by looking at the model impulse responses. These are non-linear impulse responses since the solution is not linear and thus positive and negative shocks need not to mirror themselves. Figure 13 shows the response to an impulse of ± 2 standard deviations for output growth, output level and credit spreads. As we can see, the responses are asymmetric: output growth increases by 1.8% after a positive shock while it decreases 2.2% after a negative shock of the same magnitude. In the same line, output level reacts less to the positive shock and comes back to the average value after three years while it remains below average for more than four years after a negative shock. The asymmetry is even more impressive for credit spreads: an increase of 5% after the negative shock but only a decrease of 2.6% after the positive one.

We conclude this section by arguing that the model can generate some of the asymmetries observed in data, although it fails short in generating enough asymmetry for variables like GDP growth. This is left for future research.

6 Conclusions

In this paper we started by showing three empirical facts. First, the existence of a strong statistical relation between credit spreads, our measure of financial frictions, and other typical macroeconmic variables. Second, we provided evidence that, even if many variables

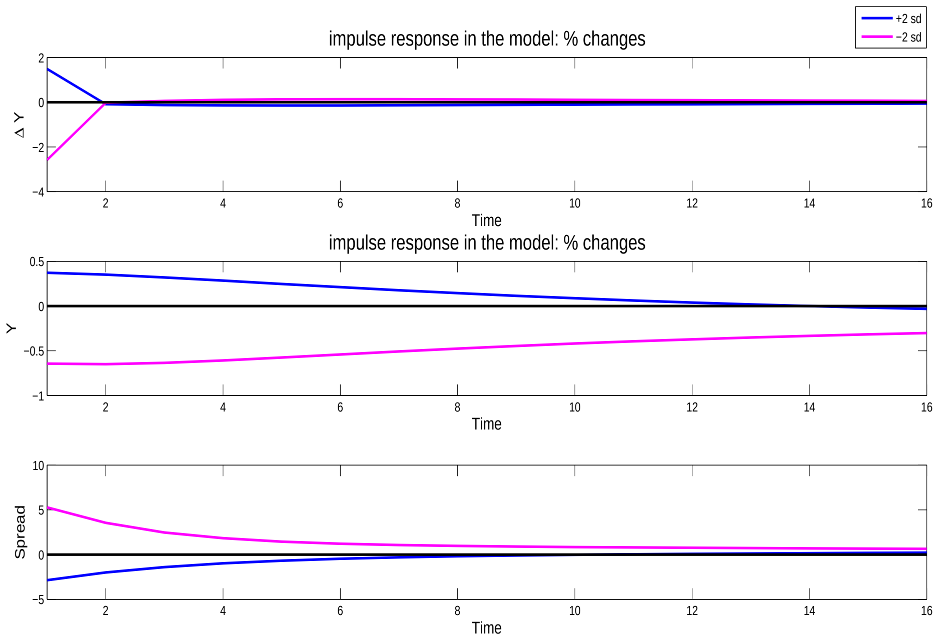

Figure 13: non-linear Impulse Responses

experiencied a “Great Moderation”, this was not te case for credit spreads. Third, we found asymmetries in the data which provides incentives for more sophisticated solution methods.

In order to tackle this, we developed a tractable model that admits an exact global solution. This is important for two reasons. First, we now have a model of default that remains simple and can thus be used for a variety of questions. Second, given the empirical findings, having model with finacial frictions that can account for non-linearities is of extreme importance. Furthermore, we showed that the model can account for many of the empirical facts. In particular, it can account for the asymmetry in credit spreads as well as the asymmetry in output responses to shocks.

At least two things are missing in the model. First, the model still can not entirely replicate the asymmetries for output, consumption and investment. Second, in this enviroment Ricardian equivalence does not hold and thus adding a government with a lending/borrowing policy seems very promising. These things are left for future research.

A Priors

As typical in the literature, we assume that hyperpameters are independent as well as independent from initial conditions.

$$f(\theta_0, h_0, \beta, \sigma, Q) = f(\theta_0) f(h_0) f(\beta)(\sigma) f(Q)$$

For the initial condition θ_0 we use a Gaussian prior

$$p(\theta_0) = \mathcal{N}(\bar{\theta}, \bar{P})$$

where $\bar{\theta}$ and $\bar{P}$ are set to the OLS estimate using a training period from 1959:1 to 1972:4[22].

We use an inverse-Wishart prior for the matrix Q which is the natural conjugate

$$f(Q) = IW(\bar{Q}, T_0)$$

In order not to impose high variability of θ_t in our priors, we set $\bar{Q}$ to a "small" number. In particular

$$\bar{Q} = (0.00001) * \bar{P}$$

However, we set the degrees of freedom T_0 as flat as possible in order to allow data to determine the posterior of Q. In particular, we set T_0 to the minimum possible such that $f(Q)$ is proper

$$T_0 = \dim(\theta_t) + 1$$

Priors for elements in the R_t matrix are in the same line: natural conjugates as uninformative as possible (large variance). In particular we set

$$\begin{aligned} f(\ln h_{i0}) &= \mathcal{N}(\ln \bar{h}_i, 10) \\ f(\beta) &= \mathcal{N}(0, 10000 \cdot I_4) \\ f(\sigma_i^2) &= IG(0.01, 1) \end{aligned}$$

[22]Since during these year the Gilchrist and Zakrajsek measure of spreads is not available we used the BAA-AAA measure of spreads

where $\ln \bar{h}_i$ is the sample variance of the variable in the training period. Both priors for β and σ have large variance in order to be very uninformative.

In general, the main idea is to impose very little weight in the priors but choose the natural conjugates in order to make the MCMC algorithm efficient. Also notice that, contrary to Cogley and Sargent (2005), no stability condition on θ_t is imposed in our priors.

After trying different priors, we found that only changes in $\bar{Q}$ seem to have impact, specially in the volatility of θ_t. However, posterior means are not dramatically affected.

B Proofs

Proof: Optimal Policies Regarding Production.

Problem of the entrepreneur is $\pi(a,k,s) = \max_n\{(axk)^{1-\alpha}n^\alpha - w(s)n\}$ From first order conditions and some simple algebra we obtain

$$n(a,k,s) = \left(\frac{\alpha}{w(s)}\right)^{\frac{1}{1-\alpha}} axk$$

Introducing this in output yields

$$y(a,k,s) = \left(\frac{\alpha}{w(s)}\right)^{\frac{\alpha}{1-\alpha}} axk$$

Finally, introducing this in the expression for profist we get

$$\pi(a,k,s) = axkw(s)^{-\frac{\alpha}{1-\alpha}}\left[\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right]$$

■

Proof: Entrepreneur's Optimal Portfoloio and Consumption Decision.

Without loss of generality write the policy for consumption as $c = (1-\varsigma)\omega$, for some function $\varsigma : R\times S \to R_+$. Then, using the budget constraint and the definition of χ' we have that

$$\begin{aligned} k' &= \frac{1}{p(\chi',s)\chi'+1}\varsigma\omega \\ b' &= \frac{\chi'}{p(\chi',s)\chi'+1}\varsigma\omega \end{aligned}$$

Also, using the law of motion for ω' we have

$$\omega' = \begin{cases} (1-\varphi)[(1-\delta)+\psi(s')a']\frac{1}{p(\chi',s)\chi'+1}\varsigma\omega & \text{if } a' < \underline{a}(\chi',s') \\ \\ \{\chi' + [(1-\delta)+\psi(s')a']\}\,\frac{1}{p(\chi',s)\chi'+1}\varsigma\omega & \text{if } a' \geq \underline{a}(\chi',s') \end{cases}$$

Next guess that $v^e(\omega,s) = A(s) + B\log(\omega)$. Introducing these expresions in (14) and some simple algebra we have

$$\begin{aligned} v^e(\omega,s) &= \max_{\{\varsigma,\chi'\}} \Bigg\{ \log(1-\varsigma) + \theta\beta\log(\varsigma) + [1+\beta\theta]\log(\omega) + \beta\mathbb{E}[A(s')|s] \\ &+ \theta\beta B\mathbb{E}\left[\int_0^{\underline{a}(\chi',s')} \log\left(\frac{(1-\varphi)[(1-\delta)+\psi(s')a]}{p(\chi',s)\chi'+1}\right) f(a)da\right] \\ &+ \theta\beta B\mathbb{E}\left[\int_{\underline{a}(\chi',s')}^{\infty} \log\left(\frac{\chi'+[(1-\delta)+\psi(s')a]}{p(\chi',s)\chi'+1}\right) f(a)da\right] \end{aligned}$$

To verify our guess we need $B = \frac{1}{1-\theta\beta}$. First order condition with respect to ς delivers $\varsigma = \theta\beta$. And the optimal choice for $\chi'(s)$ in (19) follows from the last two lines of (32). ■

Proof: Investor's Optimal Portfoloio and Consumption Decision.

Without loss of generality assume that policies are given by $c = (1-\varsigma)\omega$ and $p(\chi,s)b'(\chi) = \varsigma\gamma(\chi)\omega$ for some functions $\varsigma, \gamma(\chi) : R \times S \to R_+$. Note that the budget constraint impose that $\int \gamma(\chi)d\chi = 1$. Then, we have that

$$\omega' = \int \frac{R(\chi, b'(\chi), s')}{p(\chi,s)}\gamma(\chi)\varsigma\omega$$

Next, guess that $v^i(\omega,s) = A(s) + B\log(\omega)$. Then, with some algebra we get

$$\begin{aligned} v^i(\omega,s) &= \max_{\varsigma,\gamma(\chi)} \Bigg\{ (1+\beta B)\log(\omega) + \{\log(1-\varsigma) + \beta B\log(\varsigma)\} \\ &+ \beta\mathbb{E}\left[A(s')|s\right] + \beta B\mathbb{E}\left[\log\left(\int \frac{R(\chi,b'(\chi),s')}{p(\chi,s)}\gamma(\chi)d\chi\right)|s\right]\Bigg\} \\ \text{s. to} & \quad \int \gamma(\chi)d\chi = 1 \end{aligned}$$

Then, to verify our guess we need $B = \frac{1}{1-\beta}$. Also, taking first order conditions with respect to ς we get

$\varsigma = \beta$. Then, also we have that the optimal portfolio is given by

$$\begin{aligned} \{\gamma(\chi)\} &= \arg\max_{\{\gamma(\chi)\}} \mathbb{E}\left[\log\left(\int \frac{R(\chi, b'(\chi), s')}{p(\chi, s)}\gamma(\chi)d\chi\right)\right] \\ \text{s. to} & \quad \int \gamma(\chi)d\chi = 1 \end{aligned}$$

Finally, for $\gamma(\chi) \neq 0$ the objective function is continuous and we can use first order conditions with respect to $\gamma(\chi)$ to get

$$\begin{aligned} p(\chi, s) &= \mathbb{E}\left[R(\chi, b'(\chi), s')\frac{1}{\int \frac{R(\tilde{\chi}, b'(\tilde{\chi}), s')}{p(\tilde{\chi}, s)}\gamma(\tilde{\chi})d\tilde{\chi}}\right] \\ &= \mathbb{E}\left[R(\chi, b'(\chi), s')\frac{\beta\omega}{\int \frac{R(\tilde{\chi}, b'(\tilde{\chi}), s')}{p(\tilde{\chi}, s)}\gamma(\tilde{\chi})\beta\omega d\tilde{\chi}}\right] \\ &= \mathbb{E}\left[\beta\frac{\omega}{\omega'}R(\chi, b'(\chi), s')\right] \\ &= \mathbb{E}\left[\beta\frac{c}{c'}R(\chi, b'(\chi), s')\right] \\ &= \mathbb{E}\left[\Lambda(s, s')R(\chi, b'(\chi), s')\right] \end{aligned}$$

where the third line uses that $\omega' = \int \frac{R(\tilde{\chi}, b'(\tilde{\chi}), s')}{p(\tilde{\chi}, s)}\gamma(\tilde{\chi})\beta\omega d\tilde{\chi}$.

To conclude the proof we need

$$\begin{aligned} A(s) &= \beta\mathbb{E}[A(s')|s] + \max_{\{\gamma(\chi)\}} \frac{\beta}{1-\beta}\mathbb{E}\left[\log\left(\int \frac{R(\chi, b'(\chi), s')}{p(\chi, s)}\gamma(\chi)d\chi\right)\right] \\ \text{s. to} & \quad \int \gamma(\chi)d\chi = 1 \end{aligned}$$

which concludes the proof. ■

Proof: Equilibrium Characterization.

Integrating over bond demands and entrepreneur capital demands, we get

$$B^{e\prime}(s) = \frac{\chi'(s)}{p(\chi'(s), s)\chi'(s) + 1}\theta\beta W^e(s) \quad (32)$$

$$B^{i\prime}(s) = \frac{1}{p(\chi'(s), s)}\beta W^i(s) \quad (33)$$

$$K'(s) = \frac{1}{p(\chi'(s), s)\chi'(s) + 1}\theta\beta W^e(s) \quad (34)$$

where $W^e(s)$ and $W^i(s)$ are aggregate wealth held by entrepreneurs and investors respectively. Market clearing for bonds implies

$$p(\chi'(s), s) = -\frac{1}{\chi'(s)} \frac{W^i(s)}{W^i(s) + \theta W^e(s)}$$

which is equation (30). Using this in (33) and (34) delivers

$$\begin{aligned} B^{i\prime}(s) &= -\chi'(s)\beta[W^i(s) + \theta W^e(s)] \\ K'(s) &= \beta[W^i(s) + \theta W^e(s)] \end{aligned}$$

as is in equations (25) and (26). Wealth for the entrepreneur is simply the non-defaulted part of his bonds

$$W^i(s) = B^i[1 - F(\underline{a}(\chi(s), s))]$$

as in (27). For entrepreneurs aggregate wealth, we proceede in parts. Let $\mathbb{D} = \{i : a^i \leq \underline{a}(\chi, s)\}$, this is the set of agents who default, and $\mathbb{ND} = \mathbb{D}^c$. Then, $W^e(s) = \int_{\mathbb{D}} \omega^i di + \int_{\mathbb{ND}} \omega^i di$. We compute each term next. Note that $\int_{\mathbb{D}} di = F(\underline{a}(\chi, s))$ and $\int_{\mathbb{ND}} = 1 - F(\underline{a}(\chi, s))$. Then

$$\begin{aligned} \int_{\mathbb{D}} \omega^i di &= \int_{\mathbb{D}} (1 - \varphi)[(1 - \delta) + \psi(s)a^]k^i di \\ &= (1 - \varphi)(1 - \delta) \int_{\mathbb{D}} k^i di + (1 - \varphi)\psi(s) \int_{\mathbb{D}} a^i k^i di \\ &= (1 - \varphi)(1 - \delta)F(\underline{a}(\chi, s))K + (1 - \varphi)\psi(s) \int_{\mathbb{D}} a^i k^i di \end{aligned}$$

but

$$\begin{aligned} \int_{\mathbb{D}} a^i k^i di &= \int_0^{\underline{a}(\chi,s)} \int_0^{\infty} akd\mu(a, k) \\ &= \int_0^{\underline{a}(\chi,s)} af(a)da \int_0^{\infty} kd\mu(k) \\ &= K \int_0^{\underline{a}(\chi,s)} af(a)da \end{aligned}$$

where $\mu(a, k)$ is the joint measure of (a, k) in any particular period for this economy and the second equality follows from independence between a and k. Then

$$\int_{\mathbb{D}} \omega^i di = (1 - \varphi)(1 - \delta)F(\underline{a}(\chi, s))K + (1 - \varphi)\psi(s)K \int_0^{\underline{a}(\chi,s)} af(a)da$$

Similar algebra for $\mathbb{ND}$ yields

$$\int_{\mathbb{ND}} \omega^i di = [1 - F(\underline{a}(\chi, s))](1-\delta)K + [1 - F(\underline{a}(\chi, s))]B^e + \psi(s)K \int_{\underline{a}(\chi,s)}^{\infty} af(a)da$$

Finally,

$$\begin{aligned} W^e(s) &= [1 - \varphi F(\underline{a}(\chi, s))](1-\delta)K + [1 - F(\underline{a}(\chi, s))]B^e \\ &+ (1-\varphi)\psi(s)K \int_0^{\underline{a}(\chi,s)} af(a)da \\ &+ \psi(s)K \int_{\underline{a}(\chi,s)}^{\infty} af(a)da \end{aligned}$$

which is (28). Integrating over labor demands (6) and imposing market clearing yields

$$\begin{aligned} \bar{n} &= \int \left(\frac{\alpha}{w(s)}\right)^{\frac{1}{1-\alpha}} a^i x k^i di \\ &= \left(\frac{\alpha}{w(s)}\right)^{\frac{1}{1-\alpha}} x \int a^i k^i di \\ &= \left(\frac{\alpha}{w(s)}\right)^{\frac{1}{1-\alpha}} x\mathbb{E}(a)K \end{aligned}$$

Solving for wages yields

$$w(s) = \alpha(x\mathbb{E}(a)K)^{1-\alpha}\bar{n}^{\alpha-1} \tag{35}$$

Using the expression for wages in (10) we get

$$\psi(s) = (1-\alpha)x^{1-\alpha}\left[\mathbb{E}(a)K\right]^{-\alpha}\bar{n}^{\alpha}$$

which is equation (29). ■

C Computational Algorithm

The following steps describe the algorithm used to solve the model.

step 0: Make grids for aggregate productivity x, aggregate capital K and investors bonds B^i.

step 1: For every point in the state $s = (x, K, B^i)$ compute $\psi(s)$, $\underline{a}(\chi, s)$, $F(\underline{a}(\chi, s))$, $W^i(s)$, $W^e(s)$ and $K'(s)$ as in equations (29), (12), (27), (28) and (26) respectively. Note that all of these elements can have closed form solution.

step 2: Guess a policy $\chi'(s)$

step 3: Given the guess compute the implied policy for $B^{i\prime}(s)$ as in (25)

step 4: Given a transition probability function for x $P_x(x,x')$ and the policies for capital and bonds just obtained, computed the transition probability function for the aggregate state $P(s,s')$

$$P(s,s') = P_x(x,x')1(K'(s) \in s')1(B'(s) \in s')$$

This is done over the gird points.

step 5: Given the transition probability $P(s,s')$, solve for the implied $\hat{\chi}'(s)$ as in (19)

step 6: If $\|\hat{\chi}'(s) - \chi'(s)\| < 10^{-4}$, call this convergence. Otherwhise, update $\chi'(s)$ and go to step 3.

Using 20 points for every grid, the procedure takes in MATLAB about five hours with eight computers.

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