Notes on the Kalman Filter*

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1 Introduction

This note breifly describes the Kalman filter algorithm, how to compute the likelihood of a latent variable and how to obtain oplimal MSE inference.

2 The Kalman Filter

Assume a state-space represenation of the following form

$$z_t = A_0 + Az_{t-1} + Cw_t \tag{1}$$

$$y_t = Gz_t + v_t \tag{2}$$

Equation (1) is the state equation and equation (2) is the observation equation. We assume that z_t is $n_z \times 1$, w_t is $n_w \times 1$, y_t is $n_y \times 1$ and v_t is $n_v \times 1$. We further assume that $w_t \sim \mathcal{N}(0, I)$, $v_t \sim \mathcal{N}(0, R)$ and $\mathbb{E}(w_t v_\tau') = 0 \ \forall t, \tau$. Note that $n_v \leq n_y$.

*Preliminary and Incomplete. Please let us know if you find any mistakes: jab772@nyu.edu, gaston.navarro@nyu.edu

Let $\mathcal{I}_t$ be all the information available at time t and define first and second moment inferences as

$$\begin{aligned} z_{t|t-1} &= \mathbb{E}[z_t|\mathcal{I}_{t-1}],\ z_{t|t} = \mathbb{E}[z_t|\mathcal{I}_t], && (3) \\ \Sigma_{t|t-1} &= \mathbb{E}[(z_t - z_{t|t-1})(z_t - z_{t|t-1})'|\mathcal{I}_{t-1}] && (4) \\ \Sigma_{t|t} &= \mathbb{E}[(z_t - z_{t|t})(z_t - z_{t|t})'|\mathcal{I}_t] \end{aligned}$$

2.1 The Kalman Filter Algorithm

Four steps

(1) Choose initial conditions $z_{1|0}$ and $\Sigma_{1|0}$. A reasonable idea would be the unconditional mean and variances

$$\begin{aligned} z_{1|0} &= (I - A)^{-1} A_0 \\ \Sigma_{1|0} &= A \Sigma_{1|0} A' + CC' \end{aligned}$$

The solution for the unconditional variance is[1]

$$vec(\Sigma_{1|0}) = [I - (A \otimes A)]^{-1} vec(CC')$$

(2) Predict next period state equation

$$\begin{aligned} a_t &= y_t - G z_{t|t-1} && (5) \\ K_t &= A \Sigma_{t|t-1} G' \left(G \Sigma_{t|t-1} G' + R \right)^{-1} && (6) \\ z_{t+1|t} &= A_0 + A z_{t|t-1} + K_t a_t && (7) \\ \Sigma_{t+1|t} &= (A - K_t G) \Sigma_{t|t-1} (A - K_t G)' + CC' + K_t R K_t' && (8) \end{aligned}$$

[1]see Hamilton (1994) pg 378, chapter 13.

(3) Update inference with todays observation

$$\begin{aligned}\Omega_t &= G\Sigma_{t|t-1}G' + R && (9)\\ z_{t|t} &= z_{t|t-1} + \Sigma_{t|t-1}G'\Omega_t^{-1}a_t && (10)\\ \Sigma_{t|t} &= \Sigma_{t|t-1} - \Sigma_{t|t-1}G'\Omega_t^{-1}G\Sigma_{t|t-1} && (11)\end{aligned}$$

(4) Iterate on steps (2)-(3) for $t = 1, \ldots, T$

2.2 Estimation Step

The log-likelihood of any period is

$$\ln f(y_t|y_{t-1}, \ldots, y_1) = -\frac{n_y}{2}\ln(2\pi) - \frac{1}{2}\ln\det(\Omega_t) - \frac{1}{2}a_t'\Omega_t^{-1}a_t \tag{12}$$

Finally, the log-likelihood of the observations, givena set of parameters θ is

$$\mathcal{L}(y|\theta) = \sum_{t=1}^{T}\ln f(y_t|y_{t-1}, \ldots, y_1) \tag{13}$$

2.3 The Kalman Smoother Algorithm

If we want to "extract" the state, we may find useful to use the entire sample at each point of the inference procedure. The following algorith provides such a process

$$\begin{aligned}z_{t|T} &= z_{t|t} + \left(\Sigma_{t|t}A'\Sigma_{t+1|t}^{-1}\right)\left(z_{t+1|T} - Az_{t|t}\right) && (14)\\ \Sigma_{t|T} &= \Sigma_{t|t} - \left(\Sigma_{t|t}A'\Sigma_{t+1|t}^{-1}\right)\left(\Sigma_{t+1|T} - \Sigma_{t+1|t}\right)\left(\Sigma_{t|t}A'\Sigma_{t+1|t}^{-1}\right)' && (15)\end{aligned}$$

Note that, since $z_{T|T}$, $z_{T+1|T}$, $\Sigma_{T|T}$ and $\Sigma_{T+1|T}$ are know from the previous step, equations (14) - (15) can be computed backwards to obtain $\{z_{t|T}, \Sigma_{t|T}\}_{t=1}^{T}$.